

# Maximizing the number of edges in optimal $k$ -rankings

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## Abstract

A  $k$ -ranking is a vertex  $k$ -coloring such that if two vertices have the same color any path connecting them contains a vertex of larger color. The *rank number* of a graph is smallest  $k$  such that  $G$  has a  $k$ -ranking. For certain graphs  $G$  we consider the maximum number of edges that may be added to  $G$  without changing the rank number. Here we investigate the problem for  $G = P_{2^k-1}$ ,  $C_{2^k}$ ,  $K_{m_1, m_2, \dots, m_t}$ , and the union of two copies of  $K_n$  joined by a single edge. In addition to determining the maximum number of edges that may be added to  $G$  without changing the rank number we provide an explicit characterization of which edges change the rank number when added to  $G$ , and which edges do not.

## 1 Introduction

A *vertex coloring* of a graph is a labeling of the vertices so that no two adjacent vertices receive the same label. A  $k$ -*ranking* of a graph is a coloring of the vertex set with  $k$  positive integers such that on every path connecting two vertices of the same color there is a vertex of larger color. The *rank number of a graph* is defined to be the smallest  $k$  such that  $G$  has a  $k$ -ranking.

Early studies involving the rank number of a graph were sparked by its numerous applications including designs for very large scale integration layout (VLSI), Cholesky factorizations of matrices, and the scheduling of manufacturing systems [5, 7, 8]. Bodlaender et al. proved that given a bipartite graph  $G$  and a positive integer  $n$ , deciding whether a rank number of

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$G$  is less than  $n$  is NP-complete [2]. The rank number of paths, cycles, split graphs, complete multipartite graphs, powers of paths and cycles, and some grid graphs are well known [1, 2, 3, 4, 6, 9, 10].

In this paper we investigate an extremal property of rankings that has not yet been explored. We consider the maximum number of edges that may be added to  $G$  without changing the rank number. Since the maximum number of edges that can be added to a graph without changing the rank number varies with each particular ranking, we will focus on families where an optimal ranking has a specific structure. Here we investigate the problem for  $G = P_{2^k-1}$ ,  $C_{2^k}$ ,  $K_{m_1, m_2, \dots, m_t}$ , and the union of two copies of  $K_n$  joined by a single edge.

In addition to determining the maximum number of edges that may be added to  $G$  without changing the rank number we provide an explicit characterization of which edges change the rank number when added to  $G$ , and which edges do not. That is, given a vertex  $v_n$  in  $n$ th position in the graph, we provide an algorithm to add a new edge with  $v_n$  as one of its vertices to the graph  $G$  without changing its ranking. For this construction we use the binary representation of  $n$  to determine the position of the second vertex of the new edge. We also construct the maximum number of edges, so called *good edges*, that can be added to the graph without changing its ranking. We enumerate the maximum number of good edges.

## 2 Preliminaries

In this section we review elementary properties and known results about rankings.

A labeling  $f : V(G) \rightarrow \{1, 2, \dots, k\}$  is a  $k$ -*ranking* of a graph  $G$  if and only if  $f(u) = f(v)$  implies that every  $u - v$  path contains a vertex  $w$  such that  $f(w) > f(u)$ . Following along the lines of the chromatic number, the *rank number of a graph*  $\chi_r(G)$  is defined to be the smallest  $k$  such that  $G$  has a  $k$ -ranking. If  $H$  is a subgraph of  $G$ , then  $\chi_r(H) \leq \chi_r(G)$  (see [6]).

We use  $P_n$  to represent the path with  $n$  vertices. It is well known that a ranking of  $P_n$  with  $V(P_n) = \{v_1, v_2, \dots, v_n\}$  and  $\chi_r(P_n)$  labels can be constructed by labeling  $v_i$  with  $\alpha + 1$  where  $2^\alpha$  is the largest power of 2 that divides  $i$  [2]. We will call this ranking the *standard ranking of a path*.

We use  $C_{2^k}$  to denote a cycle with  $2^k$  vertices. A multipartite graph with  $t$  components is denoted by  $K_{m_1, m_2, \dots, m_t}$  where the  $i$ th component has  $m_i$  vertices. The complete graph with  $n$  vertices is denoted by  $K_n$ .

Let  $\Gamma$  and  $H$  be graphs with  $V(H) \subseteq V(\Gamma)$  and  $E(H) \cap E(\Gamma) = \emptyset$ . We say that an edge  $e \in H$  is *good* for  $\Gamma$  if  $\chi_r(\Gamma \cup \{e\}) = \chi_r(\Gamma)$ , and  $e$  is *forbidden* for  $\Gamma$  if  $\chi_r(\Gamma \cup \{e\}) > \chi_r(\Gamma)$ . We use  $\mu(G)$  to represent the cardinality of the maximum set of good edges for  $G$ .

For example, in Figure 1 we show a ranking of a graph  $P_{2^4-1} \cup H_P$  where  $H_P$  is the set of all good edges for  $P_{2^4-1}$ . The set of vertices of  $P_{2^4-1}$  is  $\{v_1, \dots, v_{15}\}$ . We can see that  $\chi_r(P_{2^4-1} \cup H_P) = \chi_r(P_{2^4-1}) = 4$  and that  $E(H_P)$  is comprised of 20 good edges. That is,  $\mu(P_{2^4-1}) = 20$ . Theorems 4 and 6 give necessary and sufficient conditions to determine whether

graphs  $G = P_{2^k-1} \cup H_P$  and  $P_{2^k-1}$  have the same rank number.

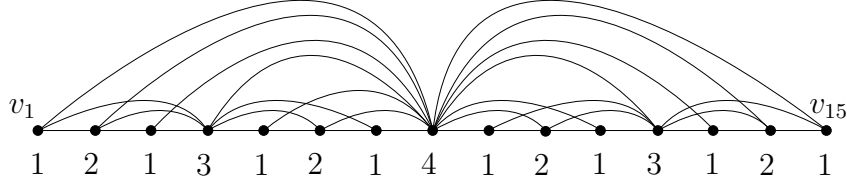


Figure 1:  $P_{2^4-1} \cup H_P$

Figure 2 Part (a) shows the graph  $G = C_{2^4} \cup H$  where  $H$  is the set of all good edges for  $C_{2^4}$ . We can see that  $\chi_r(C_{2^4} \cup H) = \chi_r(C_{2^4}) = 5$  and that  $E(H)$  is comprised of 33 good edges. That is,  $\mu(C_{2^4}) = 33$ . Theorem 8 gives necessary and sufficient conditions to determine whether the graphs  $G = C_{2^k} \cup H$  and  $C_{2^k}$  have the same rank number.

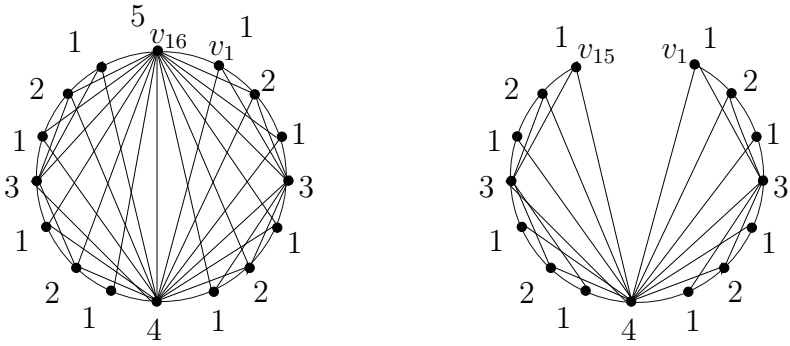


Figure 2: (a)  $G := C_{2^k} \cup H$

(b)  $G' := (C_{2^4} \setminus \{v_{16}\}) \cup H'$

**Lemma 1** ([2, 3]) *If  $k \geq 1$ , then*

1.  $P_{2^k-1}$  has a unique  $k$ -ranking and  $\chi_r(P_{2^k-1}) = k$ .
2.  $C_{2^k}$  has a unique  $k$ -ranking and  $\chi_r(C_{2^k}) = k + 1$ .

### 3 Enumeration of the Set of Good Edges for $P_{2^k-1}$

In this section we give two ways to find the maximum set of edges that may be added to  $G$  without changing the rank number. We give an algorithm to construct a good edge for  $G$ . The algorithm is based on the binary representation of  $n$ , the position of the vertex  $v_n$ . That is, given a vertex  $v_n \in G$  in  $n$ th position, the algorithm add a new edge, with  $v_n$  as one of its

vertices, to the graph  $G$  without changing its ranking. We show that if the graph  $G$  is the union of  $P_{2^t-1}$  and one edge of the form as indicated in Procedure 1, then the ranking of  $G$  is equal to the ranking of  $P_{2^t-1}$ . This guarantees that the edges constructed using Procedure 1, are good edges. We also give sufficient and necessary conditions to determine whether a set of edges  $H$  is a set of good set for  $P_{2^t-1}$ .

Since one of our aims is to enumerate the maximum number of edges that can be added to a graph without changing its rank, we give a recursive construction of the maximum set of “good edges”. The recursive construction gives us a way to count the the number of edges in the set of good edges.

We recall that  $(\alpha_r \alpha_{r-1} \dots \alpha_1 \alpha_0)_2$  with  $\alpha_h = 0$  or  $1$  for  $0 \leq h \leq r$  is the binary representation of a positive integer  $b$  if  $b = \alpha_r 2^r + \alpha_{r-1} 2^{r-1} + \dots + \alpha_1 2^1 + \alpha_0 2^0$ . We define

$$g(\alpha_i) = \begin{cases} 0 & \text{if } \alpha_i = 1 \\ 1 & \text{if } \alpha_i = 0. \end{cases}$$

**Procedure 1.** Let  $V(P_{2^k-1}) = \{v_1, v_2, \dots, v_{2^k-1}\}$  be the set of vertices of  $P_{2^k-1}$  and let  $H_P$  be a graph with  $V(H_P) = V(P_{2^k-1})$ . Suppose that  $m < n + 1$ ,  $t = \lfloor \log_2 m \rfloor$  and  $m = (\alpha_t \alpha_{t-1} \dots \alpha_1 \alpha_0)_2$ . If  $\alpha_j$  is the nonzero rightmost entry of  $m$ , then an edge  $e = \{v_m, v_n\}$  is in  $H_P$  if satisfies any of the following three conditions:

1. if  $m$  is odd then either  $n = 2^w$  for  $w > t$  or  $n = \Omega(s)$  with  $\Omega(s) = m + 1 + \sum_{i=1}^s g(\alpha_i) 2^i$  for  $s = 1, 2, \dots, t - 1$  where  $m = (\alpha_t \alpha_{t-1} \dots \alpha_1 \alpha_0)_2$ ,
2.  $m = 2^j \cdot (2l + 1)$  and  $2^j \cdot (2l + 1) + 2 \leq n < 2^j \cdot (2l + 2)$ , for  $l > 0$ ,
3.  $m = 2^j \cdot (2l + 1)$  and  $n = 2^w$  for  $2^w \geq 2^j \cdot (2l + 2)$ .

**Procedure 2.** Let  $V(C_{2^k}) = \{v_1, v_2, \dots, v_{2^k}\}$  be the set of vertices of  $C_{2^k}$ . Let  $H$  be a graph with  $V(H) = V(C_{2^k})$ . Suppose that  $m < n + 1$ ,  $t = \lfloor \log_2 m \rfloor$  and  $m = (\alpha_t \alpha_{t-1} \dots \alpha_1 \alpha_0)_2$ . If  $\alpha_j$  is the nonzero rightmost entry of  $m$ , then an edge  $e = \{v_m, v_n\}$  is in  $H$  if satisfies any of the following four conditions:

1. if  $m$  is odd then either  $n = 2^w$  for  $w > t$  or  $n = \Omega(s)$  with  $\Omega(s) = m + 1 + \sum_{i=1}^s g(\alpha_i) 2^i$  for  $s = 1, 2, \dots, t - 1$ ,
2.  $m = 2^j \cdot (2l + 1)$  and  $2^j \cdot (2l + 1) + 2 \leq n < 2^j \cdot (2l + 2)$ ,
3.  $m = 2^j \cdot (2l + 1)$  and  $n = 2^w$  for  $2^w \geq 2^j \cdot (2l + 2)$  where  $l \geq 0$ ,
4.  $1 < m < 2^k - 1$  and  $n = 2^k$ .

**Lemma 2** Suppose that  $f$  is the  $k$ -ranking of  $P_{2^k-1}$ , and  $m = (\alpha_r \alpha_{r-1} \dots \alpha_1 \alpha_0)_2$ . Let  $t = \lfloor \log_2 m \rfloor$ .

1. If  $\alpha_i = 0$  for  $i < j$  and  $\alpha_j = 1$ , then  $f(v_m) = j$ .
2.  $f(v_j) < f(v_{\Omega(i)})$  for  $m < j < \Omega(i)$  for  $i = 1, 2, \dots, t-1$ .

**Proposition 3** *Let  $e \notin P_{2^t-1}$  be an edge with vertices  $v_m$  and  $v_n$  where  $m < n$ . If  $e$  is good for  $P_{2^t-1}$  then  $e \in H_P$ .*

**Proof.** We proceed with a proof by contradiction assuming that  $e \notin H_P$ . Hence we have one of the following cases:

1.  $m$  is odd and  $2^{t+1} < n \neq 2^w$  where  $t = \lfloor \log_2 m \rfloor$ ,
2.  $m$  is odd,  $2^t < m+1 < n < 2^{t+1}$  and  $n \neq m+1 + \sum_{i=1}^s g(\alpha_i)2^i$  for  $s = 1, 2, \dots, t-1$  with  $t = \lfloor \log_2 m \rfloor$  and  $m = (\alpha_t \alpha_{t-1} \dots \alpha_1 \alpha_0)_2$ ,
3.  $m = 2^j \cdot (2l+1)$  and  $2^t < n < 2^{t+1}$  where  $2^t \geq 2^j \cdot (2l+2)$  with  $l \geq 0$ .

If Case 1 holds, then  $e$  connects vertices  $v_m$  and  $v_n$  with  $n > 2^t$ . Suppose that  $2^w < n < 2^{w+1}$ . The standard ranking  $f$  of a path implies that  $f(v_\beta) \in \{1, 2, \dots, w\}$  if  $\beta < 2^w$  and that  $f(v_\delta) \in \{1, 2, \dots, w\}$  for any  $2^w < \delta < 2^{w+1}$ . Therefore, there are two vertices  $v_\gamma$  and  $v_\rho$  such that  $f(v_\gamma) = f(v_\rho) = w$  with  $\gamma < 2^w < \rho < 2^{w+1}$ . The path containing the edge  $e$  and vertices  $v_\gamma, v_m, v_n$ , and  $v_\rho$  has two equal labels with no larger label in between, which contradicts the ranking property. Hence  $m$  is odd and  $n = 2^w$  for  $w > t$ , and  $e$  is good for  $P_{2^k-1}$ .

If Case 2 holds, then  $e$  connects vertices  $v_m$  and  $v_n$  with  $n < 2^{t+1}$ . If  $m = 2^{t+1} - 1$ , the argument is similar to the above case, so we suppose that  $m \neq 2^{t+1} - 1$ . If  $n$  is odd then  $f(v_m) = f(v_n) = 1$ , which is a contradiction.

For the remaining part of this case, we suppose that  $m+2 < n < 2^{t+1} - 1$  is even. This implies that  $m \neq 2^{t+1} - 1$  and  $m \neq 2^{t+1} - 3$  (note that the Proposition 3 is now proved for  $n < 8$ ). Therefore, there is at least one nonzero element in  $A = \{\alpha_2, \dots, \alpha_{t-1}, \alpha_t\}$  and let  $i$  be the smallest subscript such that  $\alpha_i \in A$  and  $\alpha_i = 0$ . This give rise to two subcases for the location of  $n$ :

- (a)  $n < \omega$  with  $\omega = (m+1 + g(\alpha_i)2^i)$ .
- (b)  $\Omega(r) < n < 2^{t+1}$  where  $r$  is the largest number for which the inequality holds.

If subcase (a) holds, then  $\alpha_j = 1$  for  $j < i$ . This implies that first number equal to one in the binary notation of  $m+1$  is in position  $i+1$ . This and Lemma 2 imply that  $f(v_{m+1}) = i+1$ . Therefore,  $f(v_\omega) = i+2$ , since  $m+1 < n < \omega$ . The definition of the ranking function  $f$ , implies that  $f(v_\beta) \in \{1, 2, \dots, i\}$  if  $m+1 < \beta < \omega$  and that  $f(v_\delta) \in \{1, 2, \dots, i\}$  for any  $\delta < m+1$ . Therefore, there are two vertices  $v_\gamma$  and  $v_\rho$  such that  $f(v_\gamma) = f(v_\rho) = i$  with  $\gamma < m+1 < \rho < \omega$ . The path containing the edge  $e$  and vertices  $v_\gamma, v_m, v_n$ , and  $v_\rho$  has two equal labels with no bigger label in between, which is a contradiction.

Suppose that subcase (b) holds. From Lemma 2 we know that  $f(v_d) < f(v_{\Omega(r)})$  for  $m < d < r$ , in particular we deduce that  $f(v_{\Omega(i)}) < f(v_{\Omega(r)})$  if  $i < r$ . Let  $w = f(v_{\Omega(r)})$ . This,  $P_{2^k-1}$  and the definition of  $f$  imply that  $f(v_\beta) \in \{1, 2, \dots, w-1\}$  for any  $\beta < \Omega(r)$  and that  $f(v_\delta) \in \{1, 2, \dots, w-1\}$  for any  $\Omega(r) < \delta < \Omega(r+1)$ . This implies that there are two vertices  $v_\gamma$  and  $v_\rho$  such that  $f(v_\gamma) = f(v_\rho) = w-1$  with  $\gamma < \Omega(r) < \rho < \Omega(r+1)$ . The path containing the edge  $e$  and vertices  $v_\gamma, v_m, v_n$ , and  $v_\rho$  has two equal labels with no larger label in between, which is a contradiction.

Finally suppose that Case 3 holds. That is, we suppose that every edge  $e$  connecting the vertex  $v_m$  and  $v_n$  is good, with  $m = 2^j \cdot (2l+1)$  and  $2^t < n < 2^{t+1}$  where  $2^t \geq 2^j \cdot (2l+2)$  with  $l \geq 0$ . This implies that for any  $s \leq 2^t$ , the label  $f(v_s) \in \{1, 2, \dots, t\}$ , in particular  $f(v_m) = j+1 < t$ . Since  $2^t < n < 2^{t+1}$ , the coloring  $f(v_n) \in \{1, 2, \dots, t\}$ . Then there are vertices  $v_s$  and  $v_{s'}$  with  $f(v_s) = f(v_{s'}) = t$  for  $s < 2^t$  and  $2^t < s' < 2^{t+1}$ . This is a contradiction since the path containing the edge  $e$  and connecting vertices  $v_s, v_m, v_n$  and  $v_{s'}$ , does not have a label larger than  $t$ . ■

**Theorem 4** *Let  $G = P_{2^k-1} \cup H_P$ . The set  $E(H_P)$  is the set of good edges for  $P_{2^k-1}$ .*

Theorem 4 can be proved using Proposition 3, so we omit the proof. This Theorem is equivalent to Theorem 6 Part 1. The proof of Theorem 6 counts the maximum number of good edges. We now give some definitions that are going to be used in Lemma 5 and Theorem 6. Let  $G$  be a graph with  $f$  as its  $k$ -ranking. We define  $A_j = \{v \in V(G) \mid f(v) \geq j\}$  and use  $\mathcal{C}(A_j)$  to denote the set of all component of  $G \setminus A_j$ . If  $\mathcal{C} \in \mathcal{C}(A_j)$  and  $v \in V(\mathcal{C})$ , then

$$E(v) = \{vw \mid w \in V(\mathcal{C}) \text{ and } w \text{ not adjacent to } v\}.$$

We denote by  $E_j$  the union of all sets of the form  $E(v)$  where  $f(v) = j-1$ , the vertex with maximum label in the component. The union is over all components in  $\mathcal{C}(A_j)$ . That is,

$$E_j = \bigcup_{\substack{v \in \mathcal{C}, f(v) = j-1 \\ \mathcal{C} \in \mathcal{C}(A_j)}} E(v).$$

**Lemma 5** *If  $3 < j \leq n$ , then*

1.  $P_{2^n-1} \setminus A_j$  has  $2^{n-j}$  components of the form  $P_{2^{j-1}-1}$ .
2. If  $\mathcal{C}$  is a component of  $P_{2^n-1} \setminus A_j$  and  $f(v) = j-1$  for some  $v \in \mathcal{C}$ , then  $E(v)$  is a set of good edges for  $\mathcal{C}$ .
3.  $\chi_r(P_{2^n-1} \cup E_j) = \chi_r(P_{2^n-1})$ .

**Proof.** For this proof we denote by  $f$  the  $k$ -ranking of  $P_{2^n-1}$ . We prove Part 1. Let  $u_1$  and  $u_2$  be vertices in  $P_{2^n-1}$  with  $f(u_1) \geq j$  and  $f(u_2) \geq j$  and if  $w$  is a vertex between  $u_1$  and  $u_2$  then  $f(w) < j$ . Since  $P_{2^n-1}$  has unique optimal ranking, every  $2^{j-1}$  vertices there is a vertex with label greater than or equal to  $j$  (counting from leftmost vertex). This implies that there is a path, of the form  $P_{2^{j-1}-1}$ , connecting all vertices between  $u_1$  and  $u_2$ , not including  $u_1$  and  $u_2$ . This proves that all components of  $P_{2^n-1} \setminus A_j$  are of the form  $P_{2^{j-1}-1}$ , and the total number of those components is  $\lceil (2^n - 1)/2^{j-1} \rceil = 2^{n-j+1}$ .

Proof of Part 2. Let  $v$  be a vertex in  $\mathcal{C}$  with  $f(v) = j - 1$ . Since  $j - 1$ , is the largest label in  $\mathcal{C}$ , it easy to see that every path containing edges of  $E(v)$  does not contribute to increase the ranking of the  $\mathcal{C}$ .

Proof of Part 3. Let  $G = P_{2^n-1} \cup E_j$  for some  $3 < j \leq n$ . Let  $v_1$  and  $v_2$  be vertices in  $G$  with  $f(v_1) = f(v_2)$ , suppose that both vertices are connected by a path  $P$ . Suppose  $v_1$  and  $v_2$  are in the same component  $\mathcal{C} \in \mathcal{C}(A_j)$ , then  $f(v_1) = f(v_2) < j$ . If  $P$  is a path of  $\mathcal{C}$ , then by the heredity property from  $P_{2^n-1}$ , there is a vertex in  $P$  with a label larger than  $f(v_1)$ . We now suppose that  $P$  is not a path of  $\mathcal{C}$ . Let  $v$  be the vertex in  $\mathcal{C}$  with  $f(v) = j - 1$ . These two last facts and the definition of  $E(v)$  imply that  $P$  contains an edge in  $E(v)$ . Thus, there is a vertex in  $P$  with a label larger than  $f(v_1)$ .

We suppose  $v_1$  and  $v_2$  are in different components of  $P_{2^n-1} \setminus A_j$ . So,  $f(v_1) = f(v_2) < j$ . By the definition of  $G$  and  $E_j$  we see that any path in  $G$  connecting two vertices in different component of  $P_{2^n-1} \setminus A_j$  must have at least one vertex in  $A_j$ . Since vertices in  $A_j$  have labels larger  $j - 1$ , there is a vertex in  $P$  with a label larger than  $f(v_1)$ .

We now suppose  $v_1$  and  $v_2$  are in  $A_j$ . So,  $f(v_1) = f(v_2) \geq j$ . By definition of  $k$ -ranking there is vertex  $w$  in a subpath of  $P_{2^n-1}$  that connects those two vertices, with  $f(w) > f(v_1)$ . Note that  $w \in A_j$ . Since  $w$  does not belong to any of the components in  $\mathcal{C}(A_j)$ , any other path connecting those two vertices must contain  $w$ . Therefore,  $w \in P$ . This proves Part 3. ■

**Theorem 6** *If  $n > 3$ , then*

1.  $\chi_r(P_{2^n-1} \cup \bigcup_{j=4}^n E_j) = \chi_r(P_{2^n-1}) = n$  if and only if  $\bigcup_{j=4}^n E_j$  is the set of good edges for  $P_{2^n-1}$ .
2.  $\bigcup_{j=4}^n E_j = E(H_P)$ .
3.  $\mu(P_{2^n-1}) = (n - 3)2^n + 4$ .

**Proof.** For this proof we denote by  $f$  the standard  $k$ -ranking of  $P_{2^n-1}$ . We prove of Part 1. The proof that the condition is sufficient is straightforward.

To prove that the condition is necessary we use induction. Let  $S(t)$  be the statement

$$\chi \left( P_{2^n-1} \cup \bigcup_{j=4}^t E_j \right) = \chi(P_{2^n-1}).$$

Lemma 5 Part 3. proves  $S(4)$ . Suppose the  $S(k)$  is true for some  $4 \leq k < n$ . Let

$$G_0 = P_{2^n-1} \cup \bigcup_{j=4}^k E_j \quad \text{and} \quad G_1 = P_{2^n-1} \cup \bigcup_{j=4}^{k+1} E_j.$$

Let  $v_1$  and  $v_2$  be vertices in  $G_1$  with  $f(v_1) = f(v_2)$ , and suppose that both vertices are connected by a path  $P$ . Suppose that  $v_1$  and  $v_2$  are in the same component  $\mathcal{C} \in \mathcal{C}(A_{k+1})$ , then  $f(v_1) = f(v_2) < j$ . If  $P$  is a path of  $\mathcal{C}$ , by the heredity property from  $G_0$ , there is a vertex in  $P$  with a label larger than  $f(v_1)$ .

We suppose that  $P$  is not a path of  $\mathcal{C}$ . Let  $v$  the vertex in  $\mathcal{C}$  with  $f(v) = k$ . These two last facts and definition of  $E(v)$  imply that  $P$  contains an edge in  $E(v)$ . Therefore, there is a vertex in  $P$  with a label larger than  $f(v_1)$ .

We suppose  $v_1$  and  $v_2$  are in different components of  $G_1 \setminus A_{k+1}$ . So,  $f(v_1) = f(v_2) < k + 1$ . By definition of  $G_1$  and  $E_{k+1}$  we see that any path in  $G_1$  connecting two vertices in different component of  $G_1 \setminus A_{k+1}$  has at least one vertex in  $A_{k+1}$ . Since vertices in  $A_{k+1}$  have labels larger than  $k$ , there is a vertex in  $P$  with a label larger than  $f(v_1)$ .

We now suppose that  $v_1$  and  $v_2$  are in  $A_{k+1}$ . So,  $f(v_1) = f(v_2) \geq k + 1$ . By definition of ranking there is a vertex  $w$  in a path of  $G_1$ , that connect those two vertices, with  $f(w) > f(v_1)$ . Note that  $w \in A_{k+1}$ . Since  $w$  does not belong to any of the components of  $G_1 \setminus A_j$ , any other path connecting those two vertices must contain  $w$ . Therefore,  $w \in P$ . This proves that  $S(k+1)$  is true. Thus,  $\bigcup_{j=3}^n E_j$  is a set of good edges for  $P_{2^n-1}$ .

We now prove that  $\bigcup_{i=3}^n E_i$  is the largest possible set of good edges for  $P_{2^n-1}$ . Suppose that  $uv$  is a good edge for  $P_{2^n-1}$  with  $f(v) < f(u) = j$ . If the vertices  $u$  and  $v$  are in the same component of  $P_{2^n-1} \setminus A_{j+1}$ , then is easy to see that  $uv \in E_{j+1}$ . Note that  $j$  is the largest label in each component of  $P_{2^n-1} \setminus A_{j+1}$ . If  $u$  and  $v$  are in different component of  $P_{2^n-1} \setminus A_{j+1}$ , then  $uv$  give rise to a path  $P$  connecting  $u$  and a vertex  $w$  where  $w$  and  $v$  are in the same component and  $f(w) = j$ . That is a contradiction, because  $f(u) = f(w) = j$  and  $P$  does not have label larger than  $j$ . This proves that  $\bigcup_{j=3}^n E_j$  is the set of good edges of  $P_{2^n-1}$ .

The prove of Part 2. is straightforward from Theorem 4 and Part 1.

Proof of Part 3. It easy to see that the vertex  $v_{2^n-1}$  of  $P_{2^n-1}$  has label  $n$ . That is,  $v_{2^n-1}$  is the vertex with largest color in  $P_{2^n-1}$ . Therefore,  $P_{2^n-1} \setminus A_n$  has exactly two components. Note that each component is equal to  $P_{2^{n-1}-1}$  and that the cardinality of  $E(v_{2^n-1})$  is  $(2^n - 1) - 3$ . Since  $v_{2^n-1}$  is the vertex with the largest color in  $P_{2^n-1}$ , it is easy to see (from proof of Part 1 and the proofs of Lemma 5) that the maximum number of edges that can be added to  $P_{2^n-1}$  without changing the rank is equal to the maximum number of edges that can be added to each component,  $P_{2^{n-1}-1}$ , plus all edges in  $E(v_{2^n-1})$ .

Let  $a_n = \mu(P_{2^n-1})$ . Then from the previous analysis we have that

$$a_n = 2\mu(P_{2^{n-1}-1}) + |E(v_{2^n-1})|.$$

This give rise to the recurrence relation  $a_n = 2a_{n-1} + 2^n - 4$ . Therefore, solving the recurrence relation we have that  $a_n = (n - 3)2^n + 4$ . This proves Part 3. ■



## 4 Enumeration of the Set of Good Edges for $C_{2^k}$

In this section we use the results in the previous section to find the maximum set of edges that may be added to  $C_{2^k}$  without changing the rank number (good edges).

Suppose that  $\Gamma$  represents any of the following graphs;  $C_{2^k}$ ,  $K_{m_1, m_2, \dots, m_t}$  or the graph defined by the union of two copies of  $K_n$  joined by an edge  $e$ . In this section we give sufficient and necessary conditions to determine whether a set of edges  $H$  is a good set for  $\Gamma$ . For all graphs in this section we count the number of elements in each maximum set of good edges.

We recall that in Figure 2 Part (a) we show the graph  $G = C_{2^4} \cup H$  where  $H$  is the set of all good edges for  $G$ . So,  $\chi_r(G) = \chi_r(C_{2^4}) = 5$ . In Figure 2 Part (b) we show the graph  $G' = (C_{2^k} \setminus \{v_{16}\}) \cup H'$  where  $H'$  is the set of all good edges for  $G'$ . Since the graph in Figure 2 Part (b) is equivalent to the graph in Figure 1, Theorem 6 can be applied to this graph. Theorem 8 gives sufficient and necessary conditions to determine whether the graphs  $G = C_{2^k} \cup H$  and  $C_{2^k}$  have the same rank number and counts the maximum number of good edges.

**Proposition 7** *If an edge  $e$  is good for  $P_{2^k-1}$  then  $e$  is good for  $C_{2^k}$ .*

**Proof.** Since the standard ranking of  $P_{2^k-1}$  is contained in the ranking of the cycle  $C_{2^k}$ , and the additional vertex is given the highest label, it follows that if edges are good for the path, they will be good for the cycle. ■

Let  $V := \{v_1, v_2, \dots, v_{2^k}\}$  be the set of vertices of  $C_{2^k}$ . Notice that set of edges of  $P_{2^k-1}$  is  $V \setminus \{v_{2^k}\}$ . We define

$$H_C = H_P \cup \{e \mid e \notin C_{2^k} \text{ and vertices } \{v_{2^k}, v_i\}, \text{ with } i \in \{2, \dots, 2^k - 2\}\}.$$

**Theorem 8** *If  $k > 3$ , then*

1.  $\chi_r(C_{2^k} \cup H_C) = \chi_r(C_{2^k}) = k + 1$  if and only if  $H_C$  is the set of good edges for  $C_{2^k}$ .
2.  $\mu(C_{2^n}) = (n - 2)2^n + 1$ .

**Proof.** To prove Part 1, we first show the condition is sufficient. Suppose  $\chi_r(C_{2^k} \cup H_C) = \chi_r(C_{2^k}) = k + 1$ . Suppose  $E(H)$  is not a set of good edges. Thus,  $E(H)$  contains a forbidden edge, therefore the rank number of  $C_{2^k}$  is greater than  $k + 1$ . That is a contradiction.

Next we show the condition is necessary. It is known that  $\chi_r(C_{2^k}) = k + 1$  and that this ranking is unique (up to permutation of the two largest labels) [3]. Let  $f$  be a ranking of  $C_{2^k}$  with  $k + 1$  labels where  $f(v_{2^k}) = k + 1$ .

Let  $e_1 = \{v_{2^k-1}, v_{2^k}\}$  and  $e_2 = \{v_{2^k}, v_1\}$  be two edges of  $C_{2^k}$  and let  $H'$  be the graph formed by edges of  $H$  with vertices in  $V' = V(H) \setminus \{v_t\} = \{v_1, v_2, \dots, v_{2^k-1}\}$ . Theorem 6 Parts 1 and 2 imply that  $E(H')$  is a set of good edges for the graph  $C_{2^k} \setminus \{e_1, e_2\}$  if and only

if  $\chi_r(C_{2^k} \setminus \{e_1, e_2\} \cup H') = k$  (see Figure 2 Part (b)). Note that  $V'$  is the set of vertices of  $C_{2^k} \setminus \{e_1, e_2\}$ . The vertices of  $C_{2^k} \setminus \{e_1, e_2\} \cup H'$  have same labels as vertices  $V'$ . Combining this property with  $f(v_{2^k}) = k + 1$  gives  $\chi_r(C_{2^k} \cup H') = k + 1$ .

We now prove that  $\chi_r(C_{2^k} \cup H) = k + 1$ . Let  $e$  be an edge in  $H \setminus H'$ . Therefore, the end vertices of  $e$  are  $v_{2^k}$  and  $v_n$  for some  $2 \leq n \leq 2^k - 2$ . From the ranking  $f$  of a cycle we know that  $f(v_{2^k}) = k + 1$  and  $f(v_n) < k + 1$ . Hence we do not create a new path in  $C_{2^k} \cup H_C$  connecting vertices with labels  $k + 1$ .

Proof of Part 2. Let  $W$  be the set of edges of the form  $\{v_{2^n}, v_i\}$  for  $i = 2, 3, \dots, 2^n - 2$ . The cardinality of  $W$  is  $2^n - 3$ . From Proposition 7 we know that all good edges for  $P_{2^n-1}$  are also good for  $C_{2^n}$ . Therefore, the maximum number of edges that can be added to  $C_{2^n}$  without changing the rank is equal to maximum number of edges that can be added to  $P_{2^n-1}$  plus all edges in  $W$ . Thus,  $\mu(C_{2^n}) = \mu(P_{2^n-1}) + |W|$ . This and Theorem 6 Part 3. imply that  $\mu(C_{2^n}) = (n - 3)2^n + 4 + 2^n - 3$ . Therefore,  $\mu(C_{2^n}) = (n - 2)2^n + 1$ . ■

**Theorem 9** *Let  $m_1, m_2, \dots, m_t$  be positive integers with  $m_1 = \max\{m_i\}_{i=1}^t$ . If  $G = K_{m_1, m_2, \dots, m_t}$  is a multipartite graph, then*

1. *any edge connecting two vertices in a part of order  $m_1$  is forbidden, and*
2. *any edge connecting any two vertices in any part of order  $m_i$  where  $i \neq 1$  is good.*
3.  $\mu(K_{m_1, m_2, \dots, m_t}) = \sum_{i=2}^t \frac{(m_i - 1)m_i}{2}$ .

**Proof.** Let  $W = \{w_1, w_2, \dots, w_{m_1}\}$  be the set of vertices of the part of  $G$  with order  $m_1$ . Let  $V = \{v_2, v_3, \dots, v_r\}$  be the set of vertices of  $G \setminus W$ . We consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in W \\ i & \text{if } x = v_i \text{ for some } v_i \in V. \end{cases}$$

To see that  $f$  is a minimum ranking of  $G$ , note that reducing any label violates the ranking property.

Proof of Part 1. Any edge connecting two vertices in  $W$  gives rise to a path connecting to vertices with same label.

Proof of Part 2. Any edge connecting two vertices in  $V$  does not create any new path with vertices with the same label.

Proof of Part 3. Let  $U = \{u_1, \dots, u_{m_s}\}$  the set of vertices of the part of  $K_{m_1, m_2, \dots, m_t}$  with  $m_s$  vertices and with  $s \neq 1$ . Let  $E_{m_s}$  be set of edges of the form  $\{v_i, v_j\}$  for  $i, j$  in  $\{1, 2, \dots, m_s - 2\}$  and  $j > i + 1$ . From the proof of Theorem 9 we know that  $E_{m_s}$  is a set of good edges of  $K_{m_1, m_2, \dots, m_t}$  for  $s = 2, 3, \dots, t$  (if  $s = 1$ , then  $E_{m_1}$  will be a forbidden set). The cardinality of  $E_{m_s}$  is  $(m_s - 1)m_s/2$  for  $s = 2, 3, \dots, t$ . This implies that

$$\mu(K_{m_1, m_2, \dots, m_t}) = \sum_{s=2}^t \frac{(m_s - 1)m_s}{2}.$$

This proves Part 3. ■

Let  $G_5$  be the graph defined by the union of two copies of  $K_5$  joined by an edge  $e$ . In Figure 3 we show  $G_5 \cup H$  where  $H$  is the set of all good edges for  $G_5$ . So,  $\chi_r(G_5 \cup H) = \chi_r(G_5) = 6$  and  $\mu(G_5) = 8$ . We generalize this example in Theorem 10.

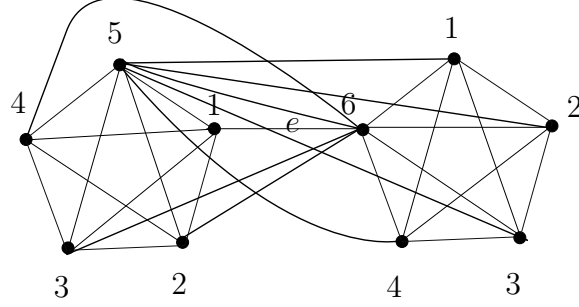


Figure 3:  $\chi_r(G_5 \cup H) = 6$

**Theorem 10** *Let  $G_n$  be the union of two copies of  $K_n$  joined by an edge. Then,*

1. *any edge connecting a vertex with highest label in one part with any other vertex in the other part is good. All other edges are forbidden. Moreover, if  $H$  is the set of all good edges for  $G_n$ , then  $\chi_r(G_n \cup H) = \chi_r(G_n) = n + 1$ .*
2.  $\mu(G_n) = 2(n - 1)$ .

**Proof.** To prove Part 1. we suppose that  $G_n = K \cup K' \cup e$  where  $K = K' = K_n$ . Let  $W = \{w_1, w_2, \dots, w_n\}$  be the set of vertices of  $K$  and let  $V = \{v_1, v_2, \dots, v_n\}$  be the set of the vertices of  $K'$  and  $\{w_1, v_n\}$  the set of vertices of  $e$ . We consider the function

$$f(x) = \begin{cases} i & \text{if } x = w_i \text{ for some } w_i \in W \\ i & \text{if } x = v_i \text{ for some } v_i \in V \setminus \{v_n\} \\ n + 1 & \text{if } x = v_n. \end{cases}$$

It is easy to see that  $f$  is a minimum ranking of  $G_n$  and that  $\chi_r(G_n) = n + 1$ . Let

$$H_1 = \{e \mid e \notin G_n \text{ is an edge with vertices } w_n, v_i \text{ for some } i \in \{1, 2, \dots, n - 1\}\}$$

and

$$H_2 = \{e \mid e \notin G_n \text{ is an edge with vertices } v_n, w_i \text{ for some } i \in \{1, 2, \dots, n - 1\}\}.$$

We prove that  $H = H_1 \cup H_2$  is the set of good edges for  $G_n$ .

From the definition of  $f$  we know the labels of the vertices in  $K$  are distinct and all of the labels in  $K'$  are distinct. The combination of these properties and the definition of  $f(v_n)$  implies that if an edge  $e$  connects one vertex in  $K$  with  $v_n$  it does not create a new edge connecting two edges with same label. Similarly, if an edge  $e$  connects one vertex in  $K'$  with  $w_n$  it does not create a path connecting two edges with the same label. This proves that  $H$  is the set of good edges for  $G_n$  and that  $\chi_r(G_n \cup H) = \chi_r(G_n) = n + 1$ .

Suppose that an edge  $e$  connects the vertices  $w_i \in K$  and  $v_j \in K'$  with  $i \leq j \neq n$ . The path  $w_j w_i v_j$  has two vertices with same label without a larger label in between. The proof of the case  $j \leq i \neq n$  is similar. Hence if  $e \notin H$ , then  $e$  is a forbidden edge.

Proof of Part 2. From proof of Part 1. we can see that  $H_1$  and  $H_2$  are the sets of good edges of  $G_n$  and that the cardinality of each set is  $n - 1$ . So,  $\mu(G_n) = |H_1| + |H_2| = 2(n - 1)$ . ■

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